Laurent polynomials

$$\mathbb{C}[t, t^{-1}] = \left\{ \sum_{i=-m}^{n} a_i t^i \;\middle|\; m, n \geq 0 \right\}$$

Q6, January 2020 (Solution by Raneeta Dutta and Neethu Suma-Raveendran.)

$(\alpha\beta)^2 \in K$ To show $[K(\sqrt{\alpha}, \sqrt{\beta}) : K] = 2$

$\alpha\beta \in K$ $[K(\sqrt{\alpha}) : K] = 2$. $x^2 - \alpha = (x \pm \sqrt{\alpha})$

$\sqrt{\alpha\beta} \in K$

$\Rightarrow \sqrt{\alpha} \cdot \sqrt{\beta} \in K \subseteq K(\sqrt{\alpha})$

$\Rightarrow \sqrt{\beta} \in K(\sqrt{\alpha})$

$$\begin{array}{c} K(\sqrt{\alpha}, \sqrt{\beta}) \\ | \\ K(\sqrt{\alpha}) \\ 2 \, | \\ K \end{array}$$

$\underbrace{[K(\sqrt{\alpha}, \sqrt{\beta}) : K]}_{2}$ = [K(sqrt(alpha), sqrt(beta)) : K(sqrt(alpha))] $\underbrace{[K(\sqrt{\alpha}) : K]}_{2}$

$$\Downarrow$$

$$[K(\sqrt{\alpha}, \sqrt{\beta}) : K(\sqrt{\alpha})] = 1$$

$$\sqrt{\beta} = a + b\sqrt{\alpha}$$

$$-a = b\sqrt{\alpha} - \sqrt{\beta} \in K$$

Q3, January 2019 (Watch the video for Aaron's explanation.)

Q6, August 2019

Treat this this like a MATH 104 problem: the factorization of a cubic of the form x^3 - a is always (x - a^{1/3})(x^2 + a^{1/3} x + a^{2/3}).

So, we need to adjoin the cube root of a as well as some complex root.

Claim: alpha is equal to the cube root of 2 and beta is equal to i (i.e., the square root of -1).

First, show that $x^3 - 2$ is irreducible over Q.

Proof. (Raneeta) It is 2-Eisenstein. (Dylan) Use the Rational Roots Theorem.

This implies that $x^3 - 2$ is the minimal polynomial of the cube root of 2.

Look at the field extension K obtained by adjoining the cube root of 2 to Q.

Next, show that $x^2 + 2^{1/3} x + 2^{2/3}$ is irreducible over K. Call its root omega.

This shows that the polynomial $x^3 - 2$ splits over the field extension L obtained by adjoining omega to K. We need to show that L is the smallest (with respect to degree) field extension in which $x^3 - 2$ splits. But it is because 2 and 3 are relatively prime.

Observation: (Raneeta) The splitting field of $x^3 - 2$ is Q($2^{1/3}$, omega), where omega is a primitive third root of unity. One can compute this.

Q3(b), August 2017

Observation: (Raneeta) We're looking for a non-separable extension.

Ex.: $\mathbb{F}_3(y^3)$ = rational function field in y^3

$\mathbb{F}_3(y)$ is a simple field extension of $\mathbb{F}_3(y^3)$.

Find the degree of F_3(y) over F_3(y^3).

But this is not a separable extension. In particular, we have that {F_3(y) : F_3(y^3)} = 1.

$$E(\mathbb{F}_3(y)/\mathbb{F}_3(y^3)) = 1$$

Find a monic irreducible polynomial f(x) with coefficients in F_3(y^3).